Boundary Regularity and Some Convergence Results for p-harmonic Functions on Metric Spaces

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Abstract:

Letbe a complete metric space equipped with a doubling Borel measure supporting the p-Poincar inequality. In this paper, we discuss and study on the regularity of p-harmonic functions at the boundary points. In particular, we show that the p-harmonic functions attain their boundary values at all regular boundary points. Moreover, the set of irregular points is a small set. We also show that the uniform limit of p-harmonic functions is p-harmonic. In addition, we obtain various convergence results for the p- harmonic functions with fixed boundary data, on an increasing sequence of open sets.

Key words: Metric space, doubling measure, Poincar inequality, p -harmonic, regularity, boundary, convergence.

الملخص:

في هذا البحث ندرس استمرارية الدوال التوافقية عند النقاط الحدية في الفضاءات المترية. بالأخص نبين أن الدوال التوافقية مستمرة عند كل النقاط الحدية المنتظمة وأن مجموعة النقاط الحدية غير المنتظمة مجموعة صغيرة. كذلك ندرس بعض مسائل التقارب حيث نبين أن التقارب المنتظم للدوال التوافقية هو أيضا دالة توافقية.

بالإضافة إلى دراسة بعض مسائل التقارب للدوال التوافقية على مجموعات مفتوحة متداخلة.

الكلمات المفتاحية: فضاء متري، متابينة بوينكر، توافقية، انتظام، حدية، تقارب.

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1.Introduction

Let $1 < p < \infty$ and $X = (X, d, \mu)$ be a complete metric space endowed with a metric d and a *doubling* Borel measure μ, i.e., there exists a constant $C \geq 1$ such that for all balls $B = B(x, r) \coloneqq \{y \in X : d(x, y) < r\}$ in X we have

$$0 < \mu(2B) < C\,\mu(B) < \infty,$$

where $2B = B(x, 2r)$.

The doubling property implies that X is complete if and only if X is proper, i.e., closed and bounded sets are compact.

In this paper we study the boundary regularity and some convergence problems for the p-harmonic functions on certain metric spaces. In R^n, the classical p-harmonic function is the solution of the p-Laplace equation, $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$, with a prescribed boundary values. An equivalent variational formulation of this problem is the minimization problem

$$\min \int |\nabla u|^p \, dx. \tag{1}$$

In a metric measure space we have no partial derivatives, i.e., no gradient but we have a substitute of the modulus for the usual gradient called *upper gradient* that was introduced by Heinonen-Koskela (1). The upper gradient enables us to define and study Sobolev type spaces in metric spaces. There are many notations of Sobolev spaces in metric spaces, see for example Cheeger (2) and Shanmugalingam (3), (4). We shall follow the definition of Shanmugalingam (3), where the Sobolev space $N^{1,p}(X)$ (called the Newtonian space) was defined as the collection of all p-integrable functions with p-integrable upper gradients, see also Farnana (5).

The p-harmonic function in metric spaces is defined to be the continuous minimizer of (1), with $|\nabla u|$ replaced by the minimal upper gradient, whose existence and uniqueness was proved in Shanmugalingam (4). The p-harmonic functions were studied e.g., in Shanmugalingam (4), Björn–Björn (6), Björn-Björn-Shanmugalingam (7) and Farnana (8).

This paper is organized as follows. In Section 2, we define Newtonian spaces, the Sobolev type spaces considered in metric spaces, and give some of their properties. We also define the p-harmonic functions with a Newtonian boundary values whose existence and uniqueness is provided by Shanmugalingam (4). It has been shown in Kinnunen-Shanmugalingam (9) that the p-harmonic functions satisfy the strong maximum principle, the Harnack's inequality and that they are locally Hölder continuous.

In Section 3, we present some convergence results for the p-harmonic functions. In particular, we show that the uniform limit of a sequence of p-harmonic functions is also p-harmonic. Moreover, we consider an increasing sequence of open sets Ω_j whose union is Ω. We analyze the convergence of the p-harmonic functions in Ω_j with fixed boundary value f.

In Section 4, we consider the p-harmonic functions with continuous boundary values. Moreover, we define and study the regular boundary points and show that the p-harmonic function attains its boundary values at all regular boundary points. Furthermore, it has been shown that most of the boundary points are regular and that the set of irregular points is a small set.

2. Notations and preliminaries

A nonnegative Borel function g on X is said to be an upper gradient of an extended real–valued function f on X if for all rectifiable curves $\gamma : [0, l\gamma] \to X$ parameterized by the arc length ds, we have

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \leq \int_\gamma g\, ds \tag{2}$$

whenever both $f\,(\gamma(0))$ and $f\,(\gamma(l\gamma))$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise. If g is a nonnegative measurable function on X and if (2) holds for p-almost every curve then g is a *p-weak upper gradient* of f.

By saying that (2) holds for p-almost every curve we mean that it fails only for a curve family with zero p-modulus, see Definition 2.1 in Shanmugalingam (3).

The upper gradient in not unique. In particular, from (2) every Borel function greater than g will be another upper gradient of f. However, if f has an upper gradient in $\boldsymbol{L}^p(X)$, then it has a unique *minimal p-weak upper gradient* $g_f \in \boldsymbol{L}^p(X)$ in the sense that for every p-weak upper gradient $g \in \boldsymbol{L}^p(X)$ of f we have $g_f \leq g$ a.e., see Corollary 3.7 in Shanmugalingam (4).

The operation of taking an upper gradient is not linear. However, we have the following useful property. If $a, b \in \boldsymbol{R}$ and g_1, g_2 are upper gradients of u_1, u_2, respectively. Then $|a|g_1 + |b|g_2$ is an upper gradient of $au_1 + bu_2$.

In Shanmugalingam (3), upper gradients have been used to define Sobolev type spaces on metric spaces. We will use the following equivalent definition.

Definition 2.1

Let $u \in \boldsymbol{L}^p(X)$, then we define

$$\| u \|_{N^{1,p}(X)} = \left(\int_X |u|^p \, d\mu + \int_X g_u^p \, d\mu \right)^{1/p}$$

where g_u is the minimal p-weak upper gradient of u. The Newtonian space on X is the quotient space

$$N^{1,p}(X) = \{u : \| u \|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\| u - v \|_{N^{1,p}(X)} = 0.$

The space $N^{1,p}(X)$ is a Banach space and a lattice, see Theorem 3.7 and p.249 in Shanmugalingam (3).

Definition 2.2

The Capacity of a set $E \subset X$ is defined by

$$C_p(E) = \inf_u \| u \|_{N^{1,p}(X)}$$

where the infimnm is taken over all $u \in N^{1,p}(X)$ such that $u \geq 1$ on E.

We say that a property holds *quasieverywhere* (q.e.) in X, if it holds everywhere except a set of capacity zero. Newtonian functions are well defined up to sets of capacity zero, i.e. if $u, v \in N^{1,p}(X)$ then $u \sim v$ if and only if $u = v$ q.e. Moreover, Corollary 3.3 in Shanmugalingam (3) shows that if $u, v \in N^{1,p}(X)$ and $u = v$ a.e., then $u = v$ q.e. in X.

From now on we assume that X supports a p–Poincaré inequality, i.e., there exist constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B(x,r)$ in X, all integrable functions u on X and all upper gradients g of u we have

$$\frac{1}{\mu(B)}\int_{B(x,r)} |u - u_{B(x,r)}|\, d\mu \leq C\, r \left(\frac{1}{\mu(B)}\int_{B(x,\lambda r)} g^p\, d\mu\right)^{1/p},$$

Under the above assumptions, every function $u \in N^{1,p}(X)$ is a *quasicontinuous*, i.e., for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $C_p(G) < \varepsilon$ and $u|_{X\setminus G}$ is continuous, see Theorem 1.1 in Björn-Björn-Shanmugalingam (10). Moreover, when restricted to R^n the Newtonian space $N^{1,p}(R^n)$ is the refined Sobolev space $W^{1,p}(R^n)$.

For $\Omega \subset X$ open we define the space $N^{1,p}(\Omega)$ with respect to the restrictions of the metric d and the measure μ to Ω. It is well known in the field that the restriction to Ω of a minimal p-weak upper gradient in X remains minimal with respect to Ω, see Björn-Björn (11).

A function u is said to belong to the *local Newtonian space* $N^{1,p}_{\mathrm{loc}}(\Omega)$ if $u \in N^{1,p}(G)$ for every $G \Subset \Omega$, where by $G \Subset \Omega$ we mean that the closure of G is a compact subset of Ω.

To be able to compare the boundary values of Newtonian functions, we need to define a Newtonian space with zero boundary values outside of Ω as follows

$$N_0^{1,p}(\Omega) = \{f|_\Omega : f \in N^{1,p}(X) \text{ and } f = 0 \text{ q.e. in } X \backslash \Omega\}.$$

Under our assumptions, Lipschitz functions with compact support are dense in $N_0^{1,p}(\Omega)$, see Shanmugalingam (4). Moreover, the proof of this result in Björn–Björn [11] shows that if $0 < u < N_0^{1,p}(\Omega)$, then we can choose the Lipschitz approximations to be nonnegative and pointwise smaller than the function u.

Definition 2.3

Suppose that $\Omega \subset X$ is open and bounded. A function $u \in N^{1,p}(\Omega)$ is a minimizer in Ω if for every function $v \in N^{1,p}(\Omega)$ with $u - v \in N_0^{1,p}(\Omega)$, we have

$$\int_\Omega g_u^p \, d\mu \leq \int_\Omega g_v^p \, d\mu,$$

where g_u and g_v are the minimal p-weak upper gradients of u and v respectively. We also say that a function u is p-harmonic if it is a continuous minimizer.

If u is a minimizer (or p-harmonic) and $\alpha, \beta \in \boldsymbol{R}$, then $\alpha u + \beta$ is a minimizer (or p-harmonic). Note, however, that the sum of two minimizers (or p-harmonic) functions need not to be a p-harmonic function and thus the theory is not linear. We instead have the minimum of two p-harmonic functions is a p-harmonic.

In Shanmugalingam (4) it was shown that there exists a unique minimizer for every $u \in N^{1,p}(\Omega)$, see also Theorem 4.2 in Farnana (8).

Theorem 2.4

Assume that Ω is open and bounded such that $C_p(X \backslash \Omega) > 0$. Let $f \in N^{1,p}(\Omega)$, then there exists a unique minimizer u (up to set of capacity zero) with $u - f \in N_0^{1,p}(\Omega)$.

Definition 2.5

By the p-harmonic extension of $f \in N^{1,p}(\Omega)$ to Ω we mean the continuous minimizer with boundary values f and will be denoted by Hf.

Note that, $-Hf$ will be the p-harmonic extension of $-f \in N^{1,p}(\Omega)$ to Ω.

Lemma 2.6 (comparison principle)

Assume that Ω is open bounded and that $C_p(X \setminus \Omega) > 0$. Let $f_1, f_2 \in N^{1,p}$(X) be such that $f_1 \leq f_2$ q.e. in $\partial\Omega$. Then $Hf_1 \leq Hf_2$ in Ω.

The p-harmonic functions satisfy many useful properties. In particular, they are locally Hölder continuous and satisfy the maximum principle: If u attains its maximum (or minimum) in Ω, then it is a constant. Moreover, nonnegative p-harmonic functions satisfy the Harnack inequality i.e., $\sup_K u \leq C \inf_K u$ for all compact $K \subset \Omega$, see Kinnunen-Shanmugalingam (9).

3. Convergence results for p-harmonic functions.

In this section, we study various convergence problems for p-harmonic functions, by letting the boundary values converge in some sense and show that the corresponding p-harmonic extensions will converge as well. Moreover, we consider an increasing sequence of open sets Ω_j whose union is Ω and fix the boundary values f. We analyze the convergence of the p-harmonic extensions in Ω_j to the p-harmonic extensions in Ω.

The following theorem shows that the uniform limit of p-harmonic functions is also p-harmonic. It is from Kinnunen- Shanmugalingam (9).

Theorem 3.1

Let $\{u_j\}_{j=1}^{\infty}$ be a sequence of p-harmonic functions that converges locally uniformly

in Ω. Then u is p-harmonic.

The following theorem is from Shanmugalingam (12).

Theorem 3.2 (Harnack's convergence theorem)

Let Ω be connected and let $\{u_j\}_{j=1}^{\infty}$ be a sequence of nonnegative p-harmonic functions in Ω. Assume that there is some $x \in \Omega$ such that $u_j(x) \leq C,\ j = 1,2,\cdots,$ for some constant C. Then a subsequence of $\{u_j\}_{j=1}^{\infty}$ converges locally uniformly to a p-harmonic in Ω.

The following result is a special case of Theorem 3.3 from Farnana (13) and also a special case of Theorem 10.18 in Björn-Björn (11).

Theorem 3.3

Let $\{f_j\}_{j=1}^{\infty}$ be a q.e. decreasing sequence such that $f_j \longrightarrow f$ in $N^{1,p}(\Omega)$ as $j \longrightarrow \infty$. Then Hf_j decreases to Hf locally uniformly in Ω.

Remark 3.4

Note that, Theorem 3.3 also holds if $\{f_j\}_{j=1}^{\infty}$ is a q.e. increasing sequence as $\{-f_j\}_{j=1}^{\infty}$ will a q.e. decreasing sequence of p-harmonic functions which implies that $-Hf_j$ decreases q.e. to $-Hf$ locally uniformly in Ω. Hence Hf_j increases q.e. to Hf locally uniformly in Ω.

In fact Theorem 3 in Kinnunen-Marola-Martio (14) shows that if $\{f_j\}_{j=1}^{\infty}$ is a bounded sequence in $N^{1,p}(\Omega)$ and $f_j \longrightarrow f$ q.e. in Ω (not necessarily monotone), as $j \longrightarrow \infty$, then $Hf_j \longrightarrow Hf$ locally uniformly in Ω, as $j \longrightarrow \infty$.

Theorem 3.5 (Corollary 9.38 in (11))

Assume that Ω is connected. Let $\{u_j\}_{j=1}^{\infty}$ be an increasing sequence of p-harmonic functions in Ω such that $u = \lim_{j\longrightarrow\infty} u_j$ is not identically ∞. Then u is p-harmonic in Ω.

Theorem 3.6 (Theorem 9.21 in (11))

Let $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \cdots \subset \Omega = \bigcup_{j=1}^{\infty} \Omega_j$. Then u is p-harmonic in Ω if and only if it is p-harmonic in Ω_j for $j = 1,2,\cdots$.

Theorem 3.7 (Theorem 9.36 in (11))

Let $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \cdots \subset \Omega = \bigcup_{j=1}^{\infty} \Omega_j$ and let u_j be p-harmonic in Ω_j, $j = 1,2,\cdots$. If $u_j \longrightarrow u$ locally uniformly in Ω, then u is p-harmonic in Ω.

4. Boundary regularity for p-harmonic functions.

In this section, we follow Björn-Björn-Shanmugalingam (7) and extend the definition of p-harmonic extension for continuous boundary functions.

The following lemma shows that Lipschitz functions on a set can be extended to a Lipschitz on a larger set, see Lemma 5.2 in Björn-Björn (11) or Theorem 6.2 in Heinonen (15).

Lemma 4.1

Let $E \subset X$ and let $f: E \longrightarrow R$ be L-Lipschitz. Then there exist two L-Lipschitz functions $\overline{f}$, $\underline{f}: X \longrightarrow R$ defined by

$$\overline{f}(x) := \inf_{y \in E}(f(y) + Ld(x,y)) \quad \text{and} \quad \underline{f}(x) := \sup_{y \in E}(f(y) + Ld(x,y)),$$

such that $\underline{f} \leq \overline{f}$ in X and that $\underline{f} = f = \overline{f}$ on E.

If $f \in \mathrm{Lip}(X) \subset N^{1,p}(X)$, then Hf is well defined. As for $f \in \mathrm{Lip}(\partial\Omega)$ the above lemma shows that f can be extended to a Lipschitz function in X. This means that, we can define the p-harmonic extension for a function $f \in \mathrm{Lip}(\partial\Omega)$ and the Hf will be independent of the choice of the extension of f, as follows: If $f \in \mathrm{Lip}(\partial\Omega)$, Lemma 4.1 shows that we can extended f to a Lipschitz function on X. If f_1, f_2 are two extensions of f then $f_1 = f_2$ on $\partial\Omega$. The comparison Lemma 2.6 shows that $Hf_1 = Hf_2$ in Ω.

If $f \in C(\partial\Omega)$, then it can be approximated uniformly by Lipschitz functions, by the Stone-Weierstrass theorem in Rudin (16), p.122

Definition 4.2

Let $f \in C(\partial\Omega)$, define $Hf : \Omega \longrightarrow \boldsymbol{R}$ by

$$Hf(x) = \sup_{\mathrm{Lip}(\partial\Omega) \ni \varphi \leq f} H\varphi(x), \qquad x \in \Omega$$

The following lemma from Björn-Björn-Shanmugalingam(7).

Lemma 4.3

Let $f \in C(\partial\Omega)$. Then Hf is p-harmonic in Ω and

$$Hf(x) = \inf_{\mathrm{Lip}(\partial\Omega) \ni \varphi \leq f} H\varphi(x) = \lim_{j\longrightarrow\infty} Hf_j(\Omega) \qquad x \in \Omega,$$

for every sequence $\{f_j\}_{j=1}^{\infty}$ of functions in $\mathrm{Lip}(\partial\Omega)$ that converges uniformly to f.

Proof.

Let $f_j \in \mathrm{Lip}(\partial\Omega)$ be such that $|f(x) - f_j(x)| < 1/j$ for all $x \in \partial\Omega$ and $j = 1,2, \ldots$. Then $|f_{j\prime}(x) - f_{j\prime\prime}(x)| \leq 2/j$ for all $x \in \partial\Omega$ and $j', j'' \geq j$. The comparison principle implies that for all $x \in \Omega$,

$$Hf_{j'}(x) - \frac{2}{j} \leq Hf_{j''}(x) \leq Hf_{j'}(x) + \frac{2}{j},$$

which shows that $\{Hf_j(x)\}_{j=1}^{\infty}$ is a Cauchy sequence in Ω. Hence the limit$h(x) := \lim_{j\longrightarrow\infty} Hf_j(x)$ exists and by Theorem 3.7 is a p-harmonic in Ω. Using the comparing principle to the inequality $f_j - \frac{1}{j} < f < f_j + \frac{1}{j}$ implies that

$$\begin{aligned} h(x) = \lim_{j\longrightarrow\infty} H(f_j - 1/j)(x) & \\ \leq \sup_{\mathrm{Lip}(\partial\Omega)\ni\varphi\leq f} H\varphi(x) &\leq \inf_{\mathrm{Lip}(\partial\Omega)\ni\varphi\leq f} H\varphi(x) \\ \leq \lim_{j\longrightarrow\infty} H(f_j + 1/j)(x) &= h(x) \end{aligned}$$

which finishes the proof.

The following lemma extends the comparison principle Lemma 2.6 to p-harmonic extensions of functions in $C(\partial\Omega)$.

Lemma 4.4 (Comparison principle).

Let Ω is bounded and that $C_p(X \setminus\Omega) > 0$. Let $f_1, f_2 \in C(\partial\Omega)$ be such that $f_1 \leq f_2$ q.e. in $\partial\Omega$. Then $Hf_1 \leq Hf_2$ in Ω.

We should mention here that the harmonic extension Hf of f to Ω is continuous only in Ω, and we may not have continuity of Hf up to the boundary. This will depend on the boundary points.

Definition 4.5

Let Ω be bounded with $C_p(X \setminus\Omega) > 0$. A point $x_0 \in \partial\Omega$ is regular if

$$\lim_{\Omega\ni x\to x_0} Hf(x) = f(x_0) \qquad \text{for all } f \in C(\partial\Omega).$$

If all $x_0 \in \Omega$ are regular, then Ω is regular. We also say that $x_0 \in \partial\Omega$ is irregular if it not regular.

The following property is from Björn-Björn-Shanmugalingam (7). It shows that the set of irregular points is a small set and that most of the boundary points are regular.

Lemma 4.6 (The Kellogg property).

Let $I_p \subset \partial\Omega$ be the set of all irregular points. Then $C_p(I_p) = 0$.

The following result shows that the p-harmonic extension of a function f attains its boundary point at all regular points.

Theorem 4.7 (Theorem 3.9 in (7))

If $f \in C(\partial\Omega)$ and $x_0 \in \partial\Omega$ is a regular boundary point, then

$$\lim_{x\to x_0} Hf(x) = f(x_0)$$

The following result from Björn-Björn-Shanmugalingam (7) shows that the p-harmonic extension of a function f is the unique p-harmonic function that attains the boundary points.

Proposition 4.8

Let $f \in C(\partial\Omega)$ assume that Ω is a regular domain. If u is p-harmonic function in Ω such that $\lim_{x\to x_0} u(x) = f(x_0)$ for all $x_0 \in \partial\Omega$. Then $u = Hf$ in Ω.

Proof

Let $\varepsilon > 0$ and for every $x_0 \in \partial\Omega$ find $r_{x_0} \in (0, \varepsilon)$ such that $|u(x) - f(x_0)| < \varepsilon$ and $|Hf(x) - f(x_0)| < \varepsilon$ whenever $x \in B(x_0, r_{x_0}) \cap \Omega$. By compactness, we have $\partial\Omega \subset \bigcup_{j=1}^{n} B_j$ with $B_j = B(x_j, r_{x_j})$. Let $\Omega' = \Omega \setminus \bigcup_{j=1}^{n} \overline{B_j}$ and choose a function $\eta \in \mathrm{Lip}_c(\Omega)$, so that $\eta = 1$ on Ω'. Then $\eta\, u \in N^{1,p}(X)$ and $\eta\, Hf \in N^{1,p}(X)$ are both p-harmonic in Ω', and furthermore, on $\partial\Omega'$ they satisfy the condition

$$\eta\, Hf - 2\varepsilon \leq \eta\, u \leq \eta\, Hf + 2\varepsilon.$$

Now the comparing principle and the fact that $\eta = 1$ on Ω' imply that

$$\eta\, Hf - 2\varepsilon \leq \eta\, u \leq \eta\, Hf + 2\varepsilon,$$

in Ω'. Letting $\varepsilon \longrightarrow 0$ we get $u = Hf$ in Ω.

Theorem 4.9

For every $f \in C(\partial\Omega)$ there exists a unique bounded p-harmonic function Hf in Ω such that

$$\lim_{\Omega \ni x \to x_0} Hf(x) = f(x_0) \qquad \text{for q.e. } x_0 \in \partial\Omega.$$

5. Conclusion

In this study, we assumed that Ω is open and bounded. We investigated the convergence of the p-harmonic functions when the boundary values vary. Moreover, we considered an increasing sequence of open sets and analyzed the convergence of the corresponding p-harmonic functions.

The findings showed that the uniform limit of p-harmonic functions is also p-harmonic. Moreover, when the boundary values converge in the $N^{1,p}(\Omega)$ space, then the corresponding p-harmonic functions converges locally uniformly in Ω. As for the boundary regularity, the study showed that for $f \in C(\partial\Omega)$ the p-harmonic extension is the unique bounded p-harmonic function that attains the boundary values at all regular points.

For future study, we recommend to study the p-harmonic functions in unbounded and non-open sets. Moreover, for the convergence problems, we recommend studying the convergence of the p-harmonic functions in the Newtonian space and under what conditions it would be obtained.

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