Fixed Points in complete 2-Metric Spaces of Orbitally Continuous Mappings

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■ **Abstract:**

This paper examines a collection of fixed-point theorems for orbitally continuous mappings in the setting of complete 2-metric spaces. Through the analysis of these mappings under various conditions, we establish results regarding the existence and uniqueness of fixed points.

Also, we provide further related results that emphasize the structural characteristics of 2-metric spaces within the framework of orbital continuity.

•**Key words**: Fixed points , complete 2- metric spaces , Orbitally continuous mappings .

■ **المستخلص:**

تتناول هذه الورقة مجموعة من مبرهنات النقطة الثابتة للتطبيقات المستمرة مدارياً ضمن إطار الفضاءات ذات المترية الثنائية الكاملة.

ومن خلال تحليل هذه التطبيقات تحت مجموعة متنوعة من الشروط، نستخلص نتائج تتعلق بوجود ووحدانية النقاط الثابتة. وايضا، نقدم نتائج أخرى ذات صلة تُبرز الخصائص البنيوية للفضاءات ذات المترية الثنائية في سياق الاستمرارية المدارية.

● **الكلمات المفتاحية** : النقاط الثابتة، الفضاءات المترية الثنائية الكاملة، الدوال المستمرة مداريًا.

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■ 1.Introduction

The concept of a 2-metric space was first introduced by S. Gähler (5) as a natural generalization of the classical notion of a metric space, where the distance function depends on three variables instead of two. Since its introduction, the theory of 2-metric spaces has attracted considerable attention and has been developed extensively across various branches of mathematics by several researchers; see, for instance, (1) , (2) and (10) . These studies have contributed significantly to the understanding of the structural properties of 2-metric spaces and have led to the establishment of numerous fundamental and interesting results in this area. In particular, the investigation of fixed point theory within the framework of 2-metric spaces has become an active and fruitful area of research. Several authors have obtained important fixed point results for different classes of mappings, including orbitally continuous mappings, as demonstrated in (3) , (4) , (6) and (7).

However, in this paper, we focus specifically on orbitally continuous mappings in the setting of complete 2-metric spaces. Our aim is to explore in greater depth the interplay between orbital continuity and the completeness structure of the space. We place special emphasis on the intrinsic structural properties of complete 2-metric spaces and analyze how these properties can be effectively utilized to establish some fixed point theorems.

2.Preliminaries

In this section, we introduce some standard definitions and preliminary results that will be used throughout the paper.

● Definition 2.1

Let F be a mapping from a non-empty set X into itself such that $Fx = x$. Then x in X is called a fixed point of F .

The Picard iteration $\{x_n\}$ in X , in the area of fixed point is given by :

$x_{n+1} = Fx_n \; (n = 0, 1, 2, ...)$,or we have $x_n = Fx_{n-1} \; (n = 1, 2, ...)$.

We construct a sequence depending on x_0 in X as follows :

$x_1 = Fx_0$

$x_2 = Fx_1 = F(Fx_0) = F^2 x_0$

$x_3 = Fx_2 = F(F^2 x_0) = F^3 x_0$

In the same way, we obtain

$x_n = F^n x_0 \quad (n = 1, 2, \ldots).$

• **Definition 2.2** (1)

Let X be a non-empty set and let $d : X \times X \times X \to \mathbb{R}$ be a mapping satisfying the following conditions :

(i) for any two distinct points $x, y \in X$ there exists a point $z \in X$ such that $d(x, y, z) \neq 0$, (ii) $d(x, y, z) = 0$ if at least two of three points x, y, z are equal , (iii) $d(x, y, z) = d(x, z, y) = d(y, z, x)$

(iv) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$, for all $x, y, z, u \in X$.

The mapping d is called a *2-metric* on X and the pair (X, d) is called a *2-metric space* .

It is worth noting that a 2-metric space is a non-negative real-valued function defined on a set with at least three elements.

• **Definition 2.3** (2)

A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be a *convergent sequence* to a point x in X if $\lim_{n \to \infty} d(x_n, x, z) = 0$ for all $z \in X$ or $d(x_n, x, z) \to 0$ as $n \to \infty$. The point x is called the *limit* of the sequence $\{x_n\}$ in X .

The limit of a sequence in a 2-metric space, if exists, is unique .

- **Definition 2.4** (2)

A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be a *Cauchy sequence* in X if $\lim_{n,m\to\infty} d(x_n, x_m, z) = 0$ for all $z \in X$, or $d(x_n, x_m, z) \to 0$ as $m, n \to \infty$.

Definition 2.5 (2)

Let (X, d) be a 2 - metric space. If every Cauchy sequence in X is convergent in X, then X is called a *complete 2-metric space*.

- **Example 2.6 (8)**

Let $(X, \|.\|)$ be a Banach space and let $x, y, z \in X$. Define

$d(x, y, z) = \|x - y\| \|y - z\| \|z - x\|$.

Then d satisfies the axioms of 2-metric function and the completeness of X follows from Banach space completeness .

Definition 2.7 (6)

Let F be a mapping of a *2* - metric space (X, d) into itself . If for all $z \in X$ $d(F^n x, u, z) \to 0$ $(n \to \infty)$ implies $d(FF^n x, Fu, z) \to 0$ $(n \to \infty)$,

then F is called an orbitally continuous mapping .

- **Lemma 2.8 (9)**

Let $\{x_n\}$ be a sequence in a complete 2 - metric space (X, d). If there exists $h \in [0,1]$ such that $d(x_n, x_{n+1}, z) \le h\, d(x_n, x_{n-1}, z)$ for all $z \in X$, then $\{x_n\}$ converges to a point in X.

3. Main Results

This section is devoted to presenting several of our findings related to the study of fixed points in complete 2-metric spaces under the framework of orbitally continuous mappings.

These results not only generalize several known theorems but also extend earlier contributions in this field.

● **Theorem 3.1**

Let F be an orbitally continuous mapping from a complete 2-metric space (X, d) into itself that satisfies the following for all $x, y, z \in X$:

$$d(Fx, Fy, z) \le a\, d(x, Fx, z) + b\, d(y, Fy, z) + c\, d(x, y, z),$$

where a, b and c are non-negative numbers such that $a + b + c < 1$.

Then F has a unique fixed point in X.

■ **Proof**

Let $x_0 \in X$. The sequence $\{x_n\}$ is given by $x_{n+1} = F x_n$ $(n = 0,1,2,\ldots)$.

We have

$$d(x_n, x_{n+1}, z) = d(Fx_{n-1}, Fx_n, z)$$

$$\le a\, d(x_{n-1}, Fx_{n-1}, z) + b\, d(x_n, Fx_n, z) +$$

$$c\, d(x_{n-1}, x_n, z)$$

$$= a\, d(x_{n-1}, x_n, z) + b\, d(x_n, x_{n+1}, z) +$$

$$c\, d(x_{n-1}, x_n, z).$$

● Therefore

$$(1 - b)\, d(x_n, x_{n+1}, z) \le (a + c)\, d(x_{n-1}, x_n, z),$$

and so $d(x_n, x_{n+1}, z) \le \dfrac{a + c}{1 - b}\, d(x_{n-1}, x_n, z)$.

Then we have $d(x_n, x_{n+1}, z) \le h\, d(x_{n-1}, x_n, z)$,

where $h = \dfrac{a + c}{1 - b} < 1$.

Hence in view of Lemma 2.8, the sequence $\{x_n\}$ converges to the element in X, say u. Thus $\lim_{n\to\infty} d(x_n, u, z) = 0$ for all $z \in X$.

We have $x_n = F^n x_0$ $(n = 1, 2, \ldots)$ which implies $\lim_{n\to\infty} d(F^n x_0, u, z) = 0 \to (1)$

Since F is orbitally continuous, so $\lim_{n\to\infty} d(FF^n x_0, Fu, z) = 0$, and hence $\lim_{n\to\infty} d(F^{n+1} x_0, Fu, z) = 0$.

Again, it follows from (1) that $\lim_{n\to\infty} d(F^{n+1} x_0, u, z) = 0$.

We have $d(u, Fu, z) \le d(u, Fu, F^{n+1}x_0) + d(u, F^{n+1}x_0, z) + d(F^{n+1}x_0, Fu, z)$ $\to 0$ $(n \to \infty)$.

Consequently, $d(u, Fu, z) = 0$. So $Fu = u$.

Thus u is a fixed point of F.

For uniqueness, let v be another fixed point of F. Then

$d(u, v, z) = d(Fu, Fv, z)$

$\le a\, d(u, Fu, z) + b\, d(v, Fv, az) + c\,(u, v, z)$.

Therefore $d(u, v, z) \le c\, d(u, v, z)$ and so $(1 - c)\, d(u, v, z) \le 0$.

Since $1 - c > 0$, it follows that $d(u, v, z) = 0$.

Thus $u = v$.

This completes the proof.

• Theorem 3.2

Let F_1 and F_2 be orbitally continuous mappings from a complete 2-metric space (X, d) into itself that satisfy the following for all $x, y, z \in X$:

$d(F_1 x, F_2 y, z) \le a\, d(x, F_1 x, z) + b\, d(y, F_2 y, z) + c\, d(x, y, z)$,

where a, b and c are non - negative numbers such that $a + b + c < 1$.

Then F_1 and F_2 have a unique common fixed point in X.

■ Proof

Let $x_0 \in X$. We define the following two sequences :

$x_{2n+1} = F_1 x_{2n}$ and $x_{2n+2} = F_2 x_{2n+1}$.

Then by routine calculations we find that

$$d(x_n, x_{n+1}, z) \le \frac{a+c}{1-b} d(x_{n-1}, x_n, z).$$

So we have $d(x_n, x_{n+1}, z) \le h\, d(x_{n-1}, x_n, z)$,

where $h = \frac{a+c}{1-b} < 1$.

Hence the sequence $\{x_n\}$ converges to the element u in X (Lemma 2.8).

Then for all $z \in X$, $\lim_{n\to\infty} d(x_n, u, z) = 0$, which implies $\lim_{n\to\infty} d(x_{2n}, u, z) = 0$.

Therefore $x_{2n} = F_1^{2n} x_0$ $(n = 1, 2, \ldots)$ and so $\lim_{n\to\infty} d(F_1^{2n} x_0, u, z) = 0$.

Since F is orbitally continuous, so $\lim_{n\to\infty} d(F_1 F_1^{2n} x_0, F_1 u, z) = 0$ and hence $\lim_{n\to\infty} d(F_1^{2n+1} x_0, F_1 u, z) = 0$.

We have $d(u, F_1 u, z) \le d(u, F_1 u, F_1^{2n+1} x_0) + d(u, F_1^{2n+1} x_0, z) + d(F_1^{2n+1} x_0, F_1 u, z) \to 0 \ (n \to \infty)$.

Then $d(u, F_1 u, z) = 0$ and so $F_1 u = u$. Thus u is a fixed point of F_1.

The uniqueness of F_1 is identical to that of F in Theorem 3.1 .

In the same way, we can obtain $F_2 u = u$.

Hence u is a unique common fixed point of F_1 and F_2.

• Corollary 3.3

Let F_k $(k = 1, 2, \ldots, n)$ be a family orbitally continuous mapping from a complete 2 - metric space (X, d) into itself. Let $F_1 F_2 \ldots F_n$ commute with every F_k $(k = 1, 2, \ldots, n)$ that satisfy the following for all $x, y, z \in X$: $d(F_1 F_2 \ldots F_n x, F_1 F_2 \ldots F_n y, z) \le a\, d(x, F_1 F_2 \ldots F_n x, z)$

$+ b\, d(y, F_1 F_2 \ldots F_n y, z) + c\, d(x, y, z)$, where a, b and c are non-negative numbers such that $a + b + c < 1$.

Then F_k have a unique common fixed point in X .

■ **Proof**.

Set $V = F_1 F_2 \ldots F_n$. Then $d(Vx, Vy, z) \le a\,d(x, Vx, z) + b\,d(y, Vy, z) + c\,d(x, y, z)$.

Then by Theorem 3.1, V has a unique common fixed point in X, say x'. So $Vx' = x'$.

Thus $V(F_k x') = F_k(V x')$ $(k = 1,2,\ldots,n)$ $= F_k(x')$.

Hence $F_k(x')$ is a fixed point of V. Since V has a unique fixed point x', it follows that $F_k(x') = x'$. Thus x' is a common fixed point of F_k .

For uniqueness, let x', x'' be fixed points of V .

Then $d(x', x'', z) = d(Vx', Vx'', z)$

$\le a\,d(x', Vx', z) + b\,d(x'', Vx'', z) + c\,d(x', x'', z)$.

It follows that $d(x', x'', z) \le c\,d(x', x'', z)$ and so $(1 - c)\,d(x', x'', z) \le 0$. Since $1 - c > 0$, it follows that $d(x', x'', z) = 0$.

So $x' = x''$.

This completes the proof.

● **Corollary 3.4**

Let F_k $(k = 1,2,\ldots n)$ be a family orbitally continuous mapping from a complete 2 - metric space (X, d) into itself. Let $F_1 F_2 \ldots F_n$ commute with every F_k $(k = 1,2,\ldots,n)$.

Suppose that there a system of positive integers $m_1, m_2, \ldots, m_n$ such that satisfy the following for all $x, y, z \in X$:

$$d(F_1^{m_1} F_2^{m_2} \ldots F_n^{m_n} x, F_1^{m_1} F_2^{m_2} \ldots F_n^{m_n} y, z) \le a_1\,d(x, F_1^{m_1} F_2^{m_2} \ldots F_n^{m_n} x, z)$$

$$+ a_2\,d(y, F_1^{m_1} F_2^{m_2} \ldots F_n^{m_n} y, z) + a_3\,d(x, y, z),$$

where a_1, a_2 and a_3 are nonnegative numbers such that $a_1 + a_2 + a_3 < 1$.

Then F_k have a unique common fixed point in X .

Proof

Let $U = F_1^{m_1} F_2^{m_2} \ldots F_n^{m_n}$.

The proof follows directly from Theorem 3.1 and Corollary 3.3 .

Conclusion

By exploiting the inherent structural properties of complete 2-metric spaces in conjunction with the notion of orbital continuity, we established sufficient conditions ensuring the existence and uniqueness of fixed points. Furthermore, several additional related results have been derived to complement and extend these findings.

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